Taking into account the expressions of the adjoint variables using conditions [Eq. (8)], we obtain the system

$$
\begin{aligned}
&C_1(a^2 - K_2) + C_2(a^2 - K_2) - C_3(\mathscr{b}^2 + K_2) = 0 \\
&C_1(2\omega a) - C_2(2\omega a) + C_4(2\omega \mathscr{b}) = 0 \\
&C_1 a(4\omega^2 - K_2 + a^2) - C_2 a(4\omega^2 - K_2 + a^2) \\
&\quad + C_4 \mathscr{b}(4\omega^2 - K_2 - \mathscr{b}^2) = \frac{2\omega K_2(C_1 + C_2 + C_3)}{\pm(\delta^2 - s_1^2)}
\end{aligned}
\tag{29}
$$

Owing to the linearity of the constants $C_i$ $(i = 1, \ldots, 4)$, we may consider only three arbitrary constants, for instance, $C_2/C_1$, $C_3/C_1$, and $C_4/C_1$. Thus, we obtain

$$
\begin{aligned}
\frac{C_2}{C_1} &= \frac{\pm a(\mathscr{b}^2 + K_2)(\delta^2 - s_1^2)^{1/2} - 2\omega K_2 s_1}{\pm a(\mathscr{b}^2 + K_2)(\delta^2 - s_1^2) + 2\omega K_2 s_1} \\
\frac{C_3}{C_1} &= \frac{\pm 2a(a^2 - K_2)(\delta^2 - s_1^2)^{1/2}}{\pm a(\mathscr{b}^2 + K_2)(\delta^2 - s_1^2)^{1/2} + 2\omega K_2 s_1} \\
\frac{C_4}{C_1} &= \frac{-4a\omega K_2 s_1}{\mathscr{b}[\pm(\mathscr{b}^2 + K_2)(\delta^2 - s_1^2)^{1/2} + 2\omega K_2 s_1]}
\end{aligned}
\tag{30}
$$

A restricted family of trajectories may be obtained by taking, for instance, $C_2/C_1 = 0$. By analyzing the expressions of the constants $\alpha_{ij}$ and $\beta_{ij}$ $(i,j = 1, \ldots, 8)$, we note that they depend on $C_1$ and the parameters $s_1$, $s_2$, and $s_4$. These may be determined from the condition $x_i(\tau) \in S^o$, which reads

$$
x_i(T) = x_i^0, \qquad (i = 1, \ldots, 4) \tag{31}
$$

Using Eqs. (15) and (20), the expression of the minimal fuel consumption becomes

$$
\begin{aligned}
V &= \gamma_1 + \gamma_2 T + \gamma_3 e^{2aT} + \gamma_4 e^{-2aT} + \gamma_5 \sin 2\mathscr{b}T \\
&\quad + \gamma_6 \cos 2\mathscr{b}T + \gamma_7 e^{aT} \sin \mathscr{b}T + \gamma_8 e^{aT} \cos \mathscr{b}T \\
&\quad + \gamma_9 e^{-aT} \sin \mathscr{b}T + \tau_{10} e^{-aT} \cos \mathscr{b}\tau
\end{aligned}
\tag{32}
$$

## Conclusions

In this analysis, we determined the optimal control of the acceleration vector and the trajectory of the interceptor such that the fuel consumption necessary to the maneuver of interception be minimal.

The acceleration program and the corresponding trajectory have been obtained in a closed form. The problem of obtaining the integration constants that appear in the equations requires solution of nonlinear algebraic equations.

The present Note furnishes the analytical formulas for the characteristics of the optimal motion in terms of the parameters of the terminal surface.

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# Point Stacking Technique for Set Matching


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## Introduction

Two sets of vectors $A$ and $B$ are equal (approximately equal) through translation if a vector $c$ can be found such that $A$ and the translate $B + c$ are equal (approximately equal) as point sets ($B + c$ = translate of $B = \{b + c$: $b$ is a member of $B\}$). In applications, $A$ is often provided as a table or catalog of known vectors, $B$ is a collection of measurements of the vectors in $A$, and $c$ is the unknown translational vector to be estimated. As pointed out in Ref. 1, false observations in $B$ can lead to erroneous estimates $c$ and result in false matching between vectors in the two sets. In this Note, a simple, dependable procedure called "point stacking" is presented that is not sensitive to false observations in $B$, provided that a minimal amount of valid $B$ data are available. The method finds useful applications in angular separation matching, phase matching (see Ref. 1), Earth point-source recognition problems, and general $n$-dimensional set matching problems. As a graphical tool, point stacking provides fast visual solutions to simple set matching problems.

## Stacking Method Theory

Consider two sets of constant, known $n$-dimensional vectors $A = \{a(1), a(2), \ldots, a(N)\}$ and $B = \{b(1), b(2), \ldots, b(M)\}$. Suppose the latter set defines a fixed translation of a particular subset of the former set; i.e., there exists an unknown $n$-dimensional vector $c$ such that $b(j) + c$ is a member of $\{a(1), \ldots, a(N)\}$ for each $j = 1, \ldots, M$. The aim of point stacking is the estimation of the translational vector $c$, as well as the vectors in $A$ that best correspond to the vectors in $B$ through translation by $c$.

To this end, consider the union $S$ defined as

$$
\begin{aligned}
S &= \bigcup_{j=1}^{M} A - [b(j) - x] \\
&= \bigcup_{i=1}^{N} \bigcup_{j=1}^{M} [a(i) - b(j) + x]
\end{aligned}
\tag{1}
$$

where $x$ is an arbitrarily chosen, fixed $n$-dimensional vector (in most applications $x = 0$ is appropriate). Thus defined, $S$ consists of $M$ translates of the set $A$, which amounts to the collection of all expressions $a(i) - b(j) + x$, $i \leq N$, $j \leq M$. Since by assumption $b(j) + c$ is a member of $A$ for each $j = 1, \ldots, M$, it follows that $x + c$ is a member of the translate $A - [b(j) - x]$ for $j = 1, \ldots, M$. In other words, $x + c$ appears exactly $M$ times in the union $S$ in Eq. (1). Assuming that

Received June 25, 1991; revision received Oct. 21 1991; accepted for publication Oct. 23, 1991. 

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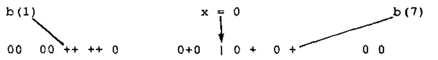


**Fig. 1 $A$ and $B$ ( O , elements of $A$; +, elements of $B$ ).**

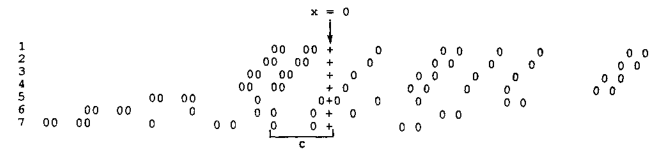


**Fig. 2 Point stacking about $x = 0$: $b(3)$ and $b(5)$ are spurious; $c$ = best offset estimate.**

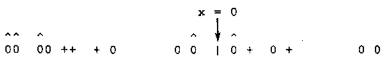


**Fig. 3 Five elements of $A$ (with circumflexes). Matching $b(1)$, $b(2)$, $b(4)$, $b(6)$, $b(7)$; spurious data $b(3)$ and $b(5)$ deleted.**

$B = \{b(1), \ldots, b(M)\}$ corresponds through translation by $c$ to a unique subset of $A$, it follows that $S$ contains one and only one subset of $M$ repeated vectors, i.e., the vector $x + c$. Using this fact, the value of $c$ is easily determined by searching through the elements of $S$ for that unique vector $v$ that appears exactly $M$ times in the union. In terms of $v$ and $x$ the value of $c$ is determined as $c = v - x$, and the $M$ vectors in $A$ that correspond, or match, the vectors in $B$ through translation by $c$ are $b(1) + c$, $b(2) + c, \ldots, b(M) + c$.

## Point Stacking with Noisy Data

In most applications the members of $A$ are known to a high degree of precision. The vectors in $B$ will ordinarily consist of noisy and/or false measurements of a subset of $A$ ($M \leq N$), or $A$ itself ($M \geq N$), where, for example, the errors associated with the $B$ data are assumed to be zero mean and Gaussian. In problems aimed at attitude estimation, or Earth point-source matching problems, the uncertainty in knowledge of $B$ will generally be greater than the uncertainty in knowledge of $A$. For example, $A$ might be a star catalog of accurately known star azimuths and $B$ a smaller set of measured azimuths taken from a star sensor onboard a spinning satellite (phase matching). In any case, the search for $M$ repeated vectors in the explanation of point stacking given earlier must be replaced by a search for a congregation or "cluster" of $M$ vectors, in order to account for the noise associated with the observations in $B$. Selection of a suitable clustering window size is discussed in the following. At present, if $v(1), \ldots, v(M)$ denote such a cluster in $S$, the sample average of the $v(i)$ becomes a sensible replacement for the repeated vector $x + c$ discussed in the general theory and reduces $x + c$ in noise-free cases.

In situations involving spurious (outlying) observations in $B$, a good matching algorithm should ignore such data in the estimation of $x + c$. Point stacking does this, provided that a minimal amount of valid $B$ data is available. The following scalar example will serve to illustrate this important feature and demonstrate the usefulness of point stacking as a visual tool in set matching problems.

Figure 1 depicts two sets, $A$ and $B$, of data referenced from 0. Plus signs indicate elements of $B$, and circles indicate elements in $A$ in the case of $N = 11$, $M = 7$. $B$ is assumed to be approximately equal to $A$ through translation by an unknown offset $c$. The aim is to estimate $c$ and the scalars in $A$ that best match the members in $B$ through translation by $c$, given that the set $B$ may contain several spurious observations. Figure 2 depicts the translated sets, $A - b(j)$, $j = 1, \ldots, 7$, which make up the union $S$ in Eq. (1) with $x = 0$. Evidently, the method of point stacking amounts to visually searching the seven translated sets of data in Fig. 2 for a maximal vertical alignment (ragged vertical stack) of circles. The maximal vertical stack contains only five circles; thus two of the observations [$b(3)$ and $b(5)$] are spurious. Using only the five valid observations leads to the estimate of the offset $c$ indicated in Fig. 2. The five corresponding matched circles from $A$ are shown in Fig. 3 using circumflexes.

In Fig. 1 the set $A$ might represent catalog star azimuths referenced from North and $B$, a set of measured azimuths sighted by a spinning satellite sensor in one sensor scan period. The sensor scans a circular strip of sky at right angles to the satellite spin axis, which points toward the Earth's center. At startup (i.e., before attitude estimation can commence) the satellite must establish a zero azimuth reference (i.e., North) from which the phase angle (angle from North, positive in the direction of spin) will be measured. Point stacking is used in Fig. 2 to estimate North using the seven-element observation set of $B$.

It is apparent that point stacking (Fig. 2) provides a fast visual solution to a somewhat complicated matching problem. The term "point stacking" arises from the fact that, if the five valid observations in Fig. 2 are vertically "stacked" about $x = 0$, then the scalars in $A$ that best match the five valid observations also form a vertical stack about the estimate $c$. In general, stacking about nonzero $x$ will result in a stack of matched vectors about $x + c$. Point stacking about nonzero $x$ may be more practical than stacking about $x = 0$ in certain visual applications where the origin 0 is not a convenient graphical reference in the viewing vicinity of the data sets $A$ and $B$.

## Cluster Radius Selection

Of crucial importance is an appropriate choice of cluster radius $EP$. By definition, $EP$ is any positive number that can be used to test members in $S$ for close clustering. If possible, $EP$ should be formulated as a function of the estimated measurement noise on $B$ and the minimum separation $d$ between points in $A$. If the noise associated with most of the $B$ data is assumed to be $N(0; \sigma^2)$ and $d$ satisfies $3\sigma < d/2$, then ideally $EP$ can be chosen as $EP = 3\sigma$. This choice will generally permit acceptable clustering and provide unique point matching between good $B$ observations and members in $A$. Reduction of the size of $A$ may be required to ensure that $d$ satisfies $3\sigma < d/2$. In azimuth phase matching, for example, $A$ can be thinned prior to point stacking by keeping only the azimuths that result from very bright stars. As a side benefit, the union $S$ is reduced in size. $S$ can further be reduced in size by keeping only the differences $a(i) - b_j$ so that the intensities corresponding to $a(i)$ and $b(j)$ are sufficiently close. With large data sets $A$ and $B$, this can dramatically reduce the size of $S$ and speed up computation.

## Point Stacking Algorithm (Pseudocode)

The following algorithm applies to $n$-dimensional point stacking. In some applications it may be advantageous to perform $n$-dimensional point stacking by looping $n$ times through a scalar version, using different values of $EP$ in each dimension. As given, the pseudocode is provided only as a guide and is aimed at finding a unique, optimal match between $B$ and $A$.

Given that $B = \{b(1), \ldots, b(M)\}$ is approximately equal to $A = \{a(1), \ldots, a(N)\}$ through translation, where $N, M > 1$. Assume that the measurement noise associated with most of the observations in $B$ is $N(0; \sigma^2)$ and that the minimum distance $d$ between the points in $A$ approximately satisfies $3\sigma < d/2$.

Compute the cluster radius: $EP = 3\sigma$.

Read in $A$ data and $B$ data.

Thin the union $S = \{a(i) - b(j)$: $a(i)$ in $A$, $b(j)$ in $B\}$, if possible, using other available information (e.g., star intensity information). Let the resulting set be $S' = \{a(i) - b(j)$: $i = 1, \ldots, N; j = 1, \ldots, M\}$.

Define the elements of $S'$ as a single array $c(k)$, $k = 1, \ldots, P$, where $P = M \cdot N$.

```
For each k = 1,...,P, determine num(k) = number of c(j)
satisfying |c(k) - c(j)| < EP.
Compute max_cluster = MAX {num(k): k = 1,...,P}.
  If (max_cluster = 1)
    No match found
    Return
  End If
  If (max_cluster > M)
    More than one match-up of B with A exists (A requires
    thinning)
    Return
  Else
    If (more than one k satisfies num(k) = max_cluster)
      More than one optimal offset c exists
      Return
    Else
      Compute c = (average of c(j), which satisfies
        |c(k) - c(j)| < EP), where k is unique and satisfies
        num(k) = max_cluster
      Output c (unique)
    End if
    Do j = 1, M
    Do i = 1, N
      If (i satisfies |a(i) - b(j) - c| < EP)
        a(i) matches b(j)
        Output matched pair [b(j), a(i)]
      End if
    End Do
    End Do
  End If
```

## Example (Phase Matching)

A visual star sensor onboard an Earth-pointing, spinning satellite is positioned at right angles to the satellite spin axis and sweeps out a 360-deg strip of sky in one sensor scan period, equal to SCAN_PERIOD seconds. Within a given scan period the sensor records $M + 1$ star sightings at times $t(1), \ldots, t(M + 1)$. The method of point stacking is used to estimate the satellite phase angle $c$ (angle of sensor from North) at time $t(1)$, as follows. A partial catalog of star azimuth values $A = \{a(1), \ldots, a(N)\}$ associated with the approximate satellite ephemeris at $t(1)$ is available, where each azimuth lies in the range [0, 360] deg. A candidate subset (superset) of rotationally equivalent angles is given by $B = \{b(1), \ldots, b(M)\}$, where $b(i) = \text{SPIN\_RATE} * [t(i + 1) - t(1)]$, and SPIN_RATE = 360/SCAN_PERIOD. Sets $A$ and $B$, together with an appropriate value of clustering window size $EP$, define the necessary algorithm inputs as outlined in the previous paragraph, where the algorithm itself has been modified to account for 360-deg angular wraparound in all instances where angles are to be compared for $EP$ closeness. The algorithm output $c$ defines the desired estimate of satellite phase angle at $t(1)$.

A specific application is afforded by selecting SCAN_PERIOD = 10 s $M = 7$, $EP = 0.01$ deg, sighting times of 1.0000, 1.0416, 1.8194, 6.0972, 7.0138, 10.4861, and 10.8194 s, and $N = 8$ catalog azimuths, given by 5, 350, 1, 345, 9, 33, 180, 220 deg. (Normally the azimuth values will be sorted, but point stacking does not require this.) The six-element $B$ set is computed as 1.4976, 29.4984, 183.4992, 216.4968, 341.4996, and 353.4984 deg. Point stacking returns the phase angle estimate $c = 3.5019$ deg at time $t(1) = 1.0000$ s and shows that the fourth and seventh star sightings at times 6.0972 and 10.8194 s are not optimally matchable with members of the given catalog set and are probably spurious.

## Conclusions

A dependable algorithm called point stacking has been developed for use as a general purpose software tool for matching two input sets of vectors. The algorithm is not sensitive to false data in either set and does not require that the sets be sorted. In effect, the algorithm estimates the most likely matchup of vectors by computing the largest possible subsets that are translationally equivalent. The method finds immediate application to satellite phase matching and Earth point-source recognition problems.

## Acknowledgments

Acknowledgment is given to Space Applications Corporation, which provided funding for this publication. The author thanks B. Stapleford for several algorithmic suggestions.

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# Orbital Motion Under Continuous Tangential Thrust

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## Introduction

THERE are relatively few problems of continuous thrusting by spacecraft in orbit for which an exact (or even a nearly exact) analytic solution for the orbital motion and mass loss is possible.[1,2] One such problem is that of optimal escape from circular orbit, which has received a good deal of attention in past years[3–6] (with notable but limited success in describing the precise motion at moderate and low levels of thrust). The pioneering work of Tsien,[3] who considered circumferential thrust in finding an approximate solution to this problem, was followed by that of Benney,[4] who considered tangential thrust, and that of Lawden,[5,6] who determined the optimum direction of thrust for minimizing expenditure of rocket propellant. It was found that, for all practical purposes, optimum thrust is tangential (or in the flight-path direction) as assumed by Benney. However, as in the solution found by Tsien, the detailed solutions for the flight path and mass loss obtained by Benney and Lawden apply only to cases of large or small thrust. Moreover, in the case of small or microthrust, the solutions obtained are valid only in the initial and intermediate portions of the escape trajectory. As noted by Lawden,[6] the assumptions required to obtain such solutions are invalid in the final portion of the trajectory as escape speed is approached and the instantaneous or osculating ellipse can no longer be considered close to a circle.

The purpose of this Note is to show how the limitations inherent in the solutions presented by Tsien, Benney, and Lawden can be eliminated if a simple change is made in the arbitrary specification of the variation of thrust magnitude. Rather than considering a constant value of tangential thrust acceleration (which is only a mathematical convenience without any practical advantage), it is assumed that the ratio of this acceleration to that of gravity is fixed. This provides a constant value of specific thrust acceleration for which the ratio of thrust to vehicle weight in orbit is fixed. The solutions

Received March 2, 1990; revision received Oct. 26, 1991; accepted for publication Feb. 20, 1992. 

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